数理解析研究所講究録
第 725 巻 1990 年 97-116

THE CONFIGURATIONS OF THE M-CURVES OF DEGREE (4,4) IN $\mathbf{RP}^1 \times \mathbf{RP}^1$ AND PERIODS OF REAL K3 SURFACES

Dedicated to Professor Haruo Suzuki on his 60th birthday

SACHIKO MATSUOKA 松岡幸子(北大・理)

Abstract. For M-curves of degree (4,4) in $\mathbf{R}P^1 \times \mathbf{R}P^1$ whose components are all contractible, it is known that three configuration types are possible. We prove that all these configuration types are realized by some M-curves of degree (4,4) by means of the existence of locally universal families of real K3 surfaces and the local surjectivity of period mappings defined over those families.

0. Introduction.

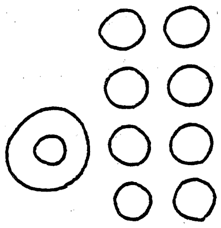
We consider the zero set $\mathbf{R}A$ of a real homogeneous polynomial F ($\neq 0$) of degree (d, r) in $\mathbf{R}P^1 \times \mathbf{R}P^1$, where d and r are integers (≥ 1). We assume that the zero set A of F in $\mathbf{C}P^1 \times \mathbf{C}P^1$ is nonsingular. (In what follows, we write $P^1 \times P^1$ for $\mathbf{C}P^1 \times \mathbf{C}P^1$.) Then A is a connected complex 1-dimensional manifold. But $\mathbf{R}A$ is a possibly disconnected real 1-dimensional manifold (a disjoint union of finitely many copies of S^1) or the empty set. It is known that the number of the connected components of $\mathbf{R}A$ does not exceed $(d-1)(r-1)+1$ (see [5]). We remark that the number $(d-1)(r-1)$ is the genus of the nonsingular curve A. We say $\mathbf{R}A$ is an *M-curve* of degree (d, r) if it has precisely $(d-1)(r-1)+1$ connected components.

In this paper we make clear the "configurations"of the M-curves of degree (4,4) in $\mathbf{R}P^1 \times \mathbf{R}P^1$, where we consider only the curves whose components (embedded S^1) are all contractible in $\mathbf{R}P^1 \times \mathbf{R}P^1$. We define the meaning of the "configurations"as follows. In our cases, each component of $\mathbf{R}A$, which is called an *oval*, divides $\mathbf{R}P^1 \times \mathbf{R}P^1$ into two connected components. One of those is homeomorphic to an open disk and called the *interior* of the oval. The other is called the *exterior* of that. As a consequence of [5], every M-curve of degree (4,4) lies in one of the following three cases.

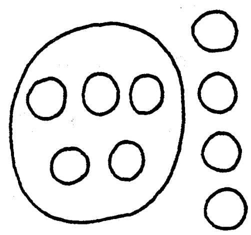
(1) Each of certain 9 ovals lies in the exteriors of the others, and the interior of one of those contains one oval. (Notation: $\frac{1}{1}8$)

(2) Each of certain 5 ovals lies in the exteriors of the others, and the interior of one of those contains 5 ovals. Each of the latter 5 ovals lies in the exteriors of the others. (Notation: $\frac{5}{1}4$)

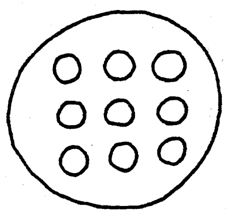
(3) An oval contains 9 ovals in its interior and each of the 9 ovals lies in the exteriors of the others. (Notation: $\frac{9}{1}$)



$\frac{1}{1}8$



$\frac{5}{1}4$



$\frac{9}{1}$

We call the above three cases the *configurations* of types $\frac{1}{1}8$, $\frac{5}{1}4$, and $\frac{9}{1}$ respectively. We can easily construct curves of degree (4,4) of configuration type $\frac{1}{1}8$ by the "Harnack's method", which is well known in the studies of Hilbert's 16th problem (see [2]). Here we omit the statement of this method. In this paper we prove that there exist curves of degree (4,4) of configuration types $\frac{5}{1}4$ and $\frac{9}{1}$ (Corollary 8 in §4). For this, it is sufficient to show the existence of *2-sheeted coverings* (for the definition, see [11]) Y of $P^1 \times P^1$ branched along nonsingular real curves of degree (4,4) whose *real parts* (see below) are homeomorphic to $\Sigma_6 \coprod 5S^2$ and $\Sigma_2 \coprod 9S^2$ respectively (see [5, §3]), where Σ_g denotes a sphere with g handles and kS^2 denotes the disjoint union of k copies of S^2. Notice that the complex conjugation of $P^1 \times P^1$ is lifted into two antiholomorphic involutions T^+ and T^- on Y. In the above statement, we call fixed point sets of these involutions real parts of Y.

It is well known that every 2-sheeted covering Y of $P^1 \times P^1$ branched along a nonsingular curve of degree (4,4) is a K3 surface. The topological types of real parts of real projective K3 surfaces are investigated in Nikulin [8]. Let h be the homology class of the preimage in Y of a hyperplane section of $P^1 \times P^1 (\subset P^3)$. Then h is *primitive* (for the definition, see [8]) in $H_2(Y, \mathbf{Z})$ and we have $h^2 = 4$. Hence the triple $(H_2(Y), T_*^{\pm}, h)$ is a *polarized integral involution* (see [8]) with invariants $\delta_L = 0, l_{(+)} = 3, l_{(-)} = 19, n = 4, t_{(+)} = 1$ and $t_{(-)}$ (for the notations, see [8]). Since we assume that $\mathbf{R}A$ is an M-curve whose components are all contractible in $\mathbf{R}P^1 \times \mathbf{R}P^1$, we moreover have $a = 0$ (see also [8]) for either T^+ or T^- because of a consequence of [5, §3]. Hence, by [8, Theorem 3.10.6], the real part of Y with respect to T^+ or T^- is homeomorphic to $\Sigma_g \coprod kS^2$, where $g = (21 - t_{(-)})/2$ and $k = (1 + t_{(-)})/2$. Furthermore, by [8, Theorem 3.4.3], a polarized integral involution with the above invariants exists if and only if $t_{(-)} = 1, 9$ or 17. By [8, Theorem 3.10.1], the isomorphism classes of polarized integral involutions with the above invariants are in bijective correspondence with the *coarse projective equivalence classes* (see [8, §3,10°]) of real projective K3 surfaces for which homology classes h of hyperplane sections (or those preimages) are primitive and $h^2 = 4$. Therefore, we see that there exist real projective K3 surfaces with $h^2 = 4$ (h: primitive) whose real parts are homeomorphic to $\Sigma_6 \coprod 5S^2$ or $\Sigma_2 \coprod 9S^2$. But these K3 surfaces are not necessarily 2-sheeted coverings of $P^1 \times P^1$ branched along nonsingular real curves of degree (4,4). We must make a closer investigation of [8, Theorem 3.10.1].

We first prepare a sufficient condition for K3 surfaces (not necessarily algebraic) with antiholomorphic involutions, which are called *real K3 surfaces*, to be 2-sheeted coverings of $P^1 \times P^1$ branched along nonsingular real curves of degree (4,4) (Lemma 2 in §2). In [3] it is proved that for every real K3 surface, there exists an "equivariant"locally universal Kähler family of its complex structures (Lemma (Kharlamov) in §1). For the real projective K3 surfaces (X, t) with $h^2 = 4$ (h: primitive) whose real parts are homeomorphic to $\Sigma_6 \coprod 5S^2$ or $\Sigma_2 \coprod 9S^2$ stated above, $L_\varphi := \mathrm{Ker}(1 + t^*)$ are isomorphic to $U \oplus U \oplus (-E_8)$ and $U \oplus U$ respectively (see [8]), where U and E_8 are even unimodular lattices with rank$U = 2$, sign$U = 0$, and rank$E_8 =$ sign$E_8 = 8$. We show that if for a real K3 surface (X, t), L_φ has $U \oplus U$ as its sublattice, then there exist real K3 surfaces which satisfy the conditions of Lemma 2 arbitrarily closely to the surface (X, t) in the equivariant family stated above (the proof of Theorem 6 in §4). Before this, we prepare Lemma 3 and its Corollary 4, which are finer versions of Tjurina's lemma concerning integer vector sequences ([**10**, Chap.IX, §5]).

The author would like to thank Professors I. Nakamura, M.-H. Saito and Y. Umezu for their kind and great help to prove Lemma 1, Professor G. Ishikawa for indicating a gap in the original proof of Theorem 6, and Professors H. Suzuki and S. Izumiya for their constant encouragement.

1. Real K3 surfaces and equivariant families of their complex structures.

We say a compact connected Kähler surface X is a *K3 surface* if the first Betti number of X vanishes and there exists a nowhere vanishing holomorphic 2-form ω_X on X. The following are known (cf.[**10**, Chap.IX]).

(1) $H^2(X,\mathbf{Z})$ is free of rank 22.

(2) The intersection form $H^2(X,\mathbf{Z}) \times H^2(X,\mathbf{Z}) \to \mathbf{Z}$ is isomorphic to $U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8)$.

(3) $\omega_X \wedge \omega_X = 0$, $\omega_X \wedge \overline{\omega}_X > 0$, $\dim_{\mathbf{C}} H^0(X,\Omega^2) = 1$. We set

$$\mathrm{Pic}X = (\omega_X)^{\perp} \cap H^2(X,\mathbf{Z}) = H^{1,1}(X) \cap H^2(X,\mathbf{Z}).$$

Since $h^1(X,O_X) = \frac{1}{2}b_1(X) = 0$, we can regard $\mathrm{Pic}X$ as the group of isomorphism classes of complex line bundles on X. We denote by $Q(\ ,\ )$ the intersection form of X. We set $P(X,\mathbf{C}) = \mathbf{P}(H^2(X,\mathbf{C}))$ and $K_{20} = \{\lambda \in P(X,\mathbf{C}) | Q(\lambda,\lambda) = 0\}$. Then we see that $H^{2,0}(X) = [\omega_X]$ is contained in K_{20}.

(4) There exists an effectively parametrized and locally universal family (V,M,π) of complex structures of X, where M is complex 20-dimensional. Here, by a family (V,M,π) of complex structures of X, we mean a C^∞-fibre bundle $\pi : V \to M$ with the fibre X, where V and M are connected complex manifolds, π is a holomorphic map onto M.

(5) For every family (V,M,π) of complex structures of a K3 surface $X = \pi^{-1}(m)$, there exists a contractible neighborhood U such that for any $\alpha \in U$, $V(\alpha) = \pi^{-1}(\alpha)$ are K3 surfaces and $(\pi^{-1}(U),U,\pi)$ is a C^∞-trivial bundle. Let $i_\alpha : V(\alpha) \to \pi^{-1}(U)$ be the

inclusion map. Then $i_\alpha^* : H^2(\pi^{-1}(U), \mathbf{Z}) \to H^2(V(\alpha), \mathbf{Z})$ is an isomorphism. We define $\tau : U \to P(X, \mathbf{C})$ by $\tau(\alpha) = i_m^* \circ i_\alpha^{*-1}(H^{2,0}(V(\alpha)))$. This is called the *period mapping.* From [**10**, Chap.IX, Theorem 2], if (V, M, π) is effectively parametrized, then τ is a holomorphic embedding on a neighbourhood U' of m in U.

Furthermore, Kharlamov [**3**] shows the following.

LEMMA (KHARLAMOV [**3**]). *Let (X, t) be a real K3 surface, namely, X is a K3 surface and t is an antiholomorphic involution on it. Then there exist a locally universal family (V, M, π) of complex structures of X and antiholomorphic involutions t_V on V and t_M on M which satisfy the following conditions.*

(i) Each fibre $V(\alpha)$ is a K3 surface and $V(m) = X$.

(ii) M is contractible, and (V, M, π) is a C^∞-trivial bundle.

(iii) τ (see (5) above) is a holomorphic embedding on M and $\tau(M)$ is a neighborhood of $\tau(m)$ in K_{20}.

(iv) $t_V|_X = t$, $\pi \circ t_V = t_M \circ \pi$, $\tau \circ t_M = \overline{t^* \circ \tau}$, where — is the natural complex conjugation on $P(X, \mathbf{C})$.

Remark. We can restrict t_V on $V(\alpha)$ for any $\alpha \in$ Fix t_M. We set $t_\alpha = t_V|_{V(\alpha)}$. Then $(V(\alpha), t_\alpha)$ are real K3 surfaces.

2. A sufficient condition for real K3 surfaces to be 2-sheeted coverings of $\mathbf{P}^1 \times \mathbf{P}^1$ branched along real curves of degree (4,4).

We prepare the following lemmas in order to catch 2-sheeted coverings (in the sense of [**11**, §1]) of $P^1 \times P^1$ branched along (real) curves in the family of (real) K3 surfaces given in §1.

LEMMA 1. *Let X be a K3 surface with rank Pic$X = 2$. If there exist primitive elements c_1 and c_2 in Pic$X$ such that $c_1{}^2 = c_2{}^2 = 0$ and $c_1 \cdot c_2 = 2$, then X can be a 2-sheeted* *branched covering of $P^1 \times P^1$, and the branch locus is a nonsingular curve of degree (4,4).*

PROOF: We choose an element b such that b and c_1 generate the free $\mathbf{Z}$-module Pic$X$. Then $c_2 = mc_1 + nb$ for some integers m and n. Since $2 = c_1 \cdot c_2 = n(c_1 \cdot b)$, we have $n = \pm 1$ or ± 2. We show that $D^2 \geq 0$ for any irreducible curve D on the surface X. In case $n = \pm 1$, we have Pic$X = \mathbf{Z}(c_1, c_2)$. Let D be an irreducible curve on X and $[D]$ be the linearly equivalence class of the divisor D. Then $[D] = kc_1 + lc_2$ for some integers k and l, and we have $D^2 = 4kl$. Since $D^2 \geq -2$, we have $D^2 \geq 0$. In case $n = \pm 2$, since c_2 is primitive, we see that m is odd. Since $(2b)^2 = (\pm c_2 \mp mc_1)^2 = -4m$, we have $b^2 = -m$. Let D be an irreducible curve on X. Then we have $[D] = kc_1 + lb$ for some integers k and l. Since $D^2 = k^2 {c_1}^2 + 2klc_1 \cdot b + l^2 b^2 = \pm 2kl - l^2 m$ and D^2 is even, we see that l is even. Hence $[D]$ is contained in $\mathbf{Z}(c_1, c_2)$. Therefore we see that $D^2 \geq 0$ as in the case $n = \pm 1$.

Now let F_i $(i = 1, 2)$ be a complex line bundle whose first Chern class is c_i. By the Riemann-Roch theorem, $h^0(F_i) + h^0(-F_i) \geq 2$. Since F_i is not trivial, we may assume that $h^0(-F_i) = 0$ and $h^0(F_i) \geq 2$ replacing c_i by $-c_i$ if necessary. We will verify that $c_1 \cdot c_2 = 2$ later on. Let C_i be the divisor of a global holomorphic section of F_i on X. We show that the complete linear system $|C_i|$ has no fixed components. If Γ is the fixed part of $|C_i|$, and D is an irreducible component of Γ, then we choose an effective divisor E such that $\Gamma + E$ is a member of $|C_i|$. We may assume that all irreducible components of E are distinct from D. In our cases, since $D^2 \geq 0$, we have $\dim|D| \geq 1$ by the Riemann-Roch theorem. Hence D is movable. This contradicts the assumption that Γ is the fixed part. Hence $|C_i|$ has no fixed components. Therefore, by [6, Proposition 1 ii)], each element of $|C_1|$ can be written as $E_1 + \cdots + E_k$ with $E_i \in |C_1'|$, C_1' being nonsingular elliptic. (For $|C_2|$, we have the same results.) Hence we have $C_1 \sim kC_1'$ (linearly equivalent). Since $[C_1'] \in \mathbf{Z}(c_1, c_2)$, we have $[C_1'] = sc_1 + tc_2$ for some integers s and t. Then, since $c_1 = k(sc_1 + tc_2)$, we see that $k = 1$. Hence we have $C_1 \sim C_1'$. Thus we may consider C_1 and C_2 to be nonsingular elliptic curves. Hence we have $C_1 \cdot C_2 = 2$. We set $C = C_1 + C_2$. The complete linear system $|C|$ also has no fixed components. Hence, by [6, Proposition 1 i)], $|C|$ has no base points and contains an irreducible nonsingular curve C'. Since $C'^2 = 4$ (> 0), the surface X is algebraic by [4, Theorem 3.3]. Thus we see that there exist elliptic curves C_1 and C_2 on the algebraic K3 surface X such that $C_1 \cdot C_2 = 2$. Then the system $|C_i|$ $(i = 1, 2)$ defines a morphism $\Phi_{|C_i|} : X \to P^1$. We can define a holomorphic mapping $\Phi : X \to P^1 \times P^1$ by the formula $\Phi(x) = (\Phi_{|C_1|}(x), \Phi_{|C_2|}(x))$ for any $x \in X$. Since $\Phi_{|C_1|}$ and $\Phi_{|C_2|}$ are surjective and $C_1 \cdot C_2 = 2$, we see that Φ is surjective. Let $S : P^1 \times P^1 \to P^3$ be the Segre embedding. This embedding gives a biholomorphic mapping onto a nonsingular quadric Q in P^3. Then the composition $S \circ \Phi : X \to P^3$ is nothing but a morphism $\Phi_{|C|}$ defined by the system $|C|$. From the well known formula $C^2 = \deg\Phi_{|C|} \cdot \deg Q$, we see that the morphism $\Phi_{|C|}$ is of degree 2. Moreover, for any irreducible curve D, the image $\Phi_{|C|}(D)$ is an irreducible curve. In fact, if $\Phi_{|C|}(D)$ is a point P, then $\Phi_{|C|}^{-1}(H) \cdot D = 0$ for a hyperplane section H of Q which does not meet the point P. Since $\Phi_{|C|}^{-1}(H)^2 = C^2 = 4$, we have $D^2 < 0$ by the Hodge index theorem. But $D^2 \geq 0$ on our surface X. This is a contradiction. We also see that for any point P in Q, the preimage $\Phi_{|C|}^{-1}(P)$ consists of finitely many points. Let B be the ramification divisor (see, for example, [**1**, p.668]) of the finite surjective mapping $\Phi_{|C|} : X \to Q$. We use the same notation B for the support of the divisor B. We set $A = \Phi_{|C|}(B)$. Then A also defines a divisor. By the definition of the ramification divisor, $\Phi_{|C|}$ is locally biholomorphic on $X \setminus B$, and in our case, all the points in B are branch points in the sense of [**11**, Definition 1.3]. Let K_X (resp. K_Q) be the canonical divisor of X (resp. Q). Then we have (see, for example, [**7**, Lemma (6.20)])

$$K_X \sim \Phi_{|C|}^*(K_Q) + B.$$

Since we know that $K_X \sim 0$ and $K_Q = (-2)(pt \times P^1 + P^1 \times pt)$ identifying Q with $P^1 \times P^1$ via the Segre embedding S, we have

$$B \sim 2\Phi^*(pt \times P^1 + P^1 \times pt).$$

Hence, in particular, $B \neq \phi$. Recall that the morphism $\Phi_{|C|}$ is of degree 2. Thus we obtain a 2-sheeted branched covering $\Phi : X \to P^1 \times P^1$ with branch locus A in the sense of [**11**, §1]. Hence the branch locus A is nonsingular. Moreover, from the proof of [**11**, Theorem 1.2], we have $[B] = \Phi^* F$ for a line bundle F over $P^1 \times P^1$ with $F^{\otimes 2} = [A]$. Since $\mathrm{Pic}(P^1 \times P^1) = \mathbf{Z}([pt \times P^1], [P^1 \times pt])$, we have $F = k[pt \times P^1] + l[P^1 \times pt]$ for some integers k and l. Since $B \sim 2\Phi^*(pt \times P^1 + P^1 \times pt)$, we have $k = l = 2$ by considering intersection numbers. Hence we have

$$A \sim 4(pt \times P^1 + P^1 \times pt).$$

Thus A is a nonsingular curve of degree (4,4). Q. E. D.

Remark. In the above lemma, for every irreducible curve D on the algebraic K3 surface X, we see that D^2 is divisible by 4. Hence, if $D^2 > 0$, then $D^2 \geq 4$, namely $p_a(D) \geq 3$. Moreover, for the irreducible curve C' ($\sim C$), we know that $p_a(C') = 3$. Hence the surface X belongs to the class $\pi = 3$ (see [**10**, Chap.VIII, p.188] or [**9**, §1, p.46]). Hence, by [**10**, Chap.VIII, Theorem 2], $\Phi_{|C|}$ is a birational morphism onto a quartic surface in P^3, or a morphism of degree 2 onto a quadric in P^3. We see that our surface X lies in the latter case.

LEMMA 2. *Let (X, t) be a real K3 surface such that X satisfies the conditions of Lemma 1. If moreover, c_1 and c_2 are contained in* Ker$(1+t^*)$, *then there exists a holomorphic mapping Φ which makes X a 2-sheeted branched covering of $P^1 \times P^1$ and satisfies $conj \circ \Phi = \Phi \circ t$. Hence the branch locus is a nonsingular curve defined by a real homogeneous polynomial of degree (4,4).*

PROOF: In the proof of Lemma 1, we define $\Phi = (\Phi_{|C_1|}, \Phi_{|C_2|})$. Let s_1 and s_2 form a basis for the space $H^0(X, O(C_1))$. Let ξ_0 and ξ_1 be holomorphic functions on X such that $\xi_1(x)s_1(x) = \xi_0(x)s_2(x)$ for any x ($\in X$). Then $\Phi_{|C_1|}$ is defined to be $[\xi_0 : \xi_1]$. We show that $conj \circ \Phi_{|C_1|} = \Phi_{|C_1|} \circ t$ if we choose an appropriate basis for $H^0(X, O(C_1))$.

We define the line bundle F_1 to be $[C_1]$. By the assumption, we see the first Chern class $c_1(F_1)$ is contained in $\mathrm{Ker}(1 + t^*)$. Hence we have $c_1(F_1) = c_1(t^*\overline{F_1})$, where $\overline{F_1}$ is the conjugate bundle of F_1. Since $H^1(X, O_X) = 0$, the line bundle F_1 and $t^*\overline{F_1}$ are isomorphic. We denote by E_1 and pr_1 the total space and the projection of F_1. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of X, $\varphi_\lambda : pr_1^{-1}(U_\lambda) \to U_\lambda \times \mathbf{C}$ be trivializations, and $g_{\lambda\mu} : U_\lambda \cap U_\mu \to \mathbf{C}^*$ be transition functions. We may assume that there exists an involution σ on Λ such that $U_{\sigma(\lambda)} = t(U_\lambda)$. Then the transition functions of the line bundle $t^*\overline{F_1}$ are $\overline{g_{\sigma(\lambda)\sigma(\mu)}} \circ t : U_\lambda \cap U_\mu \to \mathbf{C}^*$. Since F_1 and $t^*\overline{F_1}$ are isomorphic, there exists a collection of functions f_λ ($\in O^*(U_\lambda)$) such that

$$g_{\lambda\mu}(x) = \frac{f_\lambda(x)}{f_\mu(x)} \overline{g_{\sigma(\lambda)\sigma(\mu)}(t(x))} \quad \text{for any } x \ (\in U_\lambda \cap U_\mu), \tag{1}$$

where we may consider that

$$f_{\sigma(\lambda)} = \overline{f_\lambda \circ t}^{\,-1}. \tag{2}$$

Then we can define an antiholomorphic involution T_1 on E_1 such that $t \circ pr_1 = pr_1 \circ T_1$ and the restrictions $(T_1)_x : pr_1^{-1}(x) \to pr_1^{-1}(t(x))$ are antilinear as follows. (It turns out that the line bundle F_1 is a "real vector bundle ".) We define T_1 on $pr_1^{-1}(U_\lambda)$ by the following formula.

$$\varphi_{\sigma(\lambda)} \circ T_1 \circ \varphi_\lambda^{-1}(x, c) = (t(x), \overline{f_\lambda(x)^{-1}c})$$

By the equality (1), T_1 is well defined over E_1, and by (2), we see that T_1 is an involution. We now define an antilinear involution $\theta_1 : H^0(X, O(F_1)) \to H^0(X, O(F_1))$ by $\theta_1(s) = T_1 \circ s \circ t$, and choose s_1 and s_2 stated above in Fix θ_1. Then we see that $\Phi_{|C_1|} = [\overline{\xi_0 \circ t} : \overline{\xi_1 \circ t}]$. Hence $conj \circ \Phi_{|C_1|} = \Phi_{|C_1|} \circ t$. We have the same results for $|C_2|$. Thus we have $conj \circ \Phi = \Phi \circ t$. It follows that $conj(A) = A$, where A is the branch locus. Q. E. D.

3. A lemma concerning integer vector sequences.

LEMMA 3. *For any integer sequence $\alpha_1'(n)$ with $\alpha_1'(n) \to \infty$, any positive real number α, any real numbers x_3 and x_4, there exist a subsequence $\alpha_1(n)$ of $\alpha_1'(n)$ and an integer vector sequence $(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n))$ which satisfy the following five conditions.*

(1) $\beta_1\beta_2 + \beta_3\beta_4 = 1$

(2) $\lim_{n\to\infty} \dfrac{\beta_3}{\beta_1} = x_3$

(3) $\lim_{n\to\infty} \dfrac{\beta_4}{\beta_1} = x_4$

(4) *β_1 and β_4 are odd.*

(5) $\lim_{n\to\infty} \frac{\beta_1}{\alpha_1} = \alpha$

PROOF: We first prove in the case x_4 is a rational number. The rational number x_4 can be expanded into a finite simple continued fraction as follows.

$$x_4 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{r-1} + \cfrac{1}{a_r}}}}$$

In the above, a_1 is an integer, and $a_2, \dots, a_r$ are positive integers. We define (u_0, v_0), $\dots$,(u_r, v_r) inductively as follows.

$$(u_0, v_0) = (-1, -1)$$
$$(u_j, v_j) = \begin{cases} (v_{j-1}, u_{j-1}) \text{ if } a_j \text{ is even or } (u_{j-1}, v_{j-1}) = (-1, 1) \\ (v_{j-1}, -u_{j-1}) \text{ otherwise} \end{cases}$$

In the case $r \geq 2$, we define b_i $(2 \leq i \leq r)$ as follows.

$$b_i = a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{a_{i+2} + \ddots + \cfrac{1}{a_{r-1} + \cfrac{1}{a_r}}}}$$

Remark that every b_i is positive. We set $\alpha' = \frac{\alpha}{b_2 \times \cdots \times b_r}$. In the case $r = 1$, we set $\alpha' = \alpha$. Now we choose and fix a subsequence $\alpha_1(n)$ of $\alpha_1'(n)$ such that $\frac{\alpha_1(n)}{n} \to \infty$. Let $\tilde{\beta}_1(n)$ be the closest integer to $\alpha_1(n)\alpha'$. Since $\alpha_1(n) \to \infty$, we have $\lim \frac{\tilde{\beta}_1}{\alpha_1} = \alpha'$ and $\frac{\tilde{\beta}_1}{2n} = \frac{\tilde{\beta}_1}{\alpha_1}\frac{\alpha_1}{2n} \to \infty$. We set $\beta_1(n) = \left[\frac{\tilde{\beta}_1(n)}{2n}\right]$ or $\left[\frac{\tilde{\beta}_1(n)}{2n}\right] + 1$, where we take $\beta_1(n)$ to be odd (resp. even) if $v_r = -1$ (resp. 1). We have $\beta_1(n) \to \infty$. We set $x_3' = (-1)^r x_3$. In the case $(u_r, v_r) = (1, -1)$, let β_3 be the closest integer to $\beta_1 x'_3$ that is relatively prime to β_1. Since β_1 is odd, β_1 and $2\beta_3$ are relatively prime, and hence, there exist integers u and v such that $u\beta_1 + 2v\beta_3 = 1$ and $|u| < |2\beta_3|$, $|v| < |\beta_1|$. We set $\beta_2 = u$ and $\beta_4 = 2v$. In the case $(u_r, v_r) = (-1, 1)$, let β_3 be as above. Then there exist integers u and v such that $u\beta_1 + v\beta_3 = 1$ and $|u| < |\beta_3|$, $|v| < |\beta_1|$. We set $\beta_2 = u$ and $\beta_4 = v$. In the case $(u_r, v_r) = (-1, -1)$, let β_3 be the closest integer to $\beta_1 x'_3$ that is relatively prime to $2\beta_1$. Then there exist integers u and v such that $2u\beta_1 + v\beta_3 = 1$ and $|u| < |\beta_3|$, $|v| < |2\beta_1|$. We set $\beta_2 = 2u$ and $\beta_4 = v$. The case $(u_r, v_r) = (1, 1)$ cannot occur. It follows that β_4 is odd (resp. even) if $u_r = -1$ (resp. 1). In all the cases, we have $\beta_1\beta_2 + \beta_3\beta_4 = 1$, $\lim_{n\to\infty} \frac{\beta_3}{\beta_1} = x'_3$, and $|\frac{\beta_4}{\beta_1}| < 2$. We see that $\frac{\beta_2}{\beta_1}$ are also bounded. We define a new sequence $P(n) = (p_1(n), p_2(n), p_3(n), p_4(n))$ to be

$$(-\beta_4(n) + 2n\beta_1(n), -\beta_3(n), 2n\beta_3(n) + \beta_2(n), \beta_1(n)).$$

Then we have $p_1p_2 + p_3p_4 = 1$, $\lim \frac{p_3}{p_1} = x'_3$ and $\lim \frac{p_4}{p_1} = 0$. Since $|\beta_1 - \frac{\tilde{\beta}_1}{2n}| \leq 1$, $\lim \frac{\tilde{\beta}_1}{\alpha_1} = \alpha'$, and $\frac{\alpha_1}{n} \to \infty$, we have $\lim \frac{p_1}{\alpha_1} = \alpha'$. Remark that the parity of (p_1, p_2, p_3, p_4) corresponds to $(\beta_4, \beta_3, \beta_2, \beta_1)$.

We now assume that a new sequence $\beta(n) = (\beta_1, \beta_2, \beta_3, \beta_4)$ satisfies the conditions (1), (2), (3) and (5) in the statement of Lemma 3 for a positive real number α, real numbers x_3 and x_4, and a sequence $\alpha_1(n)$ with $\alpha_1(n) \to \infty$. Let k be an arbitrary integer with $k - x_4 > 0$. We define a new sequence $I_k(\beta(n)) = (q_1, q_2, q_3, q_4)$ to be

$$(-\beta_4(n) + k\beta_1(n), -\beta_3(n), k\beta_3(n) + \beta_2(n), \beta_1(n)).$$

Then we see that $q_1q_2 + q_3q_4 = 1$ and $\lim \frac{q_3}{q_1} = x_3$. Hence the properties (1) and (2) are preserved by the transformation I_k. On the other hand, we see that

$$\lim \frac{q_4}{q_1} = \frac{1}{k - x_4}$$

and

$$\lim \frac{q_1}{\alpha_1} = \alpha(k - x_4) \ (> 0).$$

We next define a new sequence $J(\beta(n))$ to be $(\beta_1, \beta_2, -\beta_3, -\beta_4)$. Then the properties (1) and (5) are preserved by the transformation J. But for the properties (2) and (3), the limit values are multiplied by (-1).

The sequence $P(n)$ has the properties (1), (2) (for $x_3 = x'_3$), (3) (for $x_4 = 0$) and (5). In the case $r \geq 2$, we can transform $P(n)$ by I_{a_r}. Then $I_{a_r}(P(n))$ has the properties (3) (for $x_4 = \frac{1}{a_r}$) and (5) (for $\alpha = \alpha' a_r = \frac{\alpha}{b_2 \times \cdots \times b_{r-1}}$ (> 0)). Next we can transform $J \circ I_{a_r}(P(n))$ by $I_{a_{r-1}}$. Then $I_{a_{r-1}} \circ J \circ I_{a_r}(P(n))$ has the properties (3) (for $x_4 = \dfrac{1}{a_{r-1} + \dfrac{1}{a_r}}$) and (5) (for $\alpha = \alpha' a_r (a_{r-1} + \dfrac{1}{a_r}) = \dfrac{\alpha}{b_2 \times \cdots \times b_{r-2}}$ (> 0)). Thus we obtain the sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) := J \circ I_{a_2} \circ J \circ \cdots \circ J \circ I_{a_{r-2}} \circ J \circ I_{a_{r-1}} \circ J \circ I_{a_r}(P(n))$. In the case $r = 1$, we set $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = P(n)$. Then we have (1) $\gamma_1\gamma_2 + \gamma_3\gamma_4 = 1$ (2) $\lim \dfrac{\gamma_3}{\gamma_1} = -x_3$ (3) $\lim \dfrac{\gamma_4}{\gamma_1} = a_1 - x_4$ (5) $\lim \dfrac{\gamma_1}{\alpha_1} = \alpha$. Finally we set $(\beta_1, \beta_2, \beta_3, \beta_4) = (\gamma_1, a_1\gamma_3 + \gamma_2, -\gamma_3, -\gamma_4 + a_1\gamma_1)$. Then this sequence satisfies the conditions (1), (2), (3) and (5) of Lemma 3. From the definition of (u_r, v_r), we observe that the condition (4) is also satisfied. Thus Lemma 3 is proved in the case x_4 is a rational number. To complete the proof of the lemma, let x_4 be an arbitrary real number. Let $\{x_4(n)\}$ $(n = 1, 2, 3 \ldots)$ be a rational number sequence which converges to x_4 satisfying $|x_4(n) - x_4| < \dfrac{1}{n}$. From the results above, there exist sequences $(\beta_{1n}, \beta_{2n}, \beta_{3n}, \beta_{4n})$ such that (1) $\beta_{1n}\beta_{2n} + \beta_{3n}\beta_{4n} = 1$ (2)$\lim_{m\to\infty} \dfrac{\beta_{3n}(m)}{\beta_{1n}(m)} = x_3$ (3)$\lim_{m\to\infty} \dfrac{\beta_{4n}(m)}{\beta_{1n}(m)} = x_4(n)$ (4)β_{1n} and β_{4n} are odd. (5)$\lim_{m\to\infty} \dfrac{\beta_{1n}(m)}{\alpha_1(m)} = \alpha$. Remark that the subsequence $\alpha_1(m)$ of $\alpha_1'(m)$ does not depend on n. We choose a natural number sequence $m(1) < m(2) < m(3) < \cdots$ such that $|\dfrac{\beta_{3n}(m(n))}{\beta_{1n}(m(n))} - x_3| < \dfrac{1}{n}$, $|\dfrac{\beta_{4n}(m(n))}{\beta_{1n}(m(n))} - x_4(n)| < \dfrac{1}{n}$ and $|\dfrac{\beta_{1n}(m(n))}{\alpha_1(m(n))} - \alpha| < \dfrac{1}{n}$. We set $(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n)) = (\beta_1(m(n)), \beta_2(m(n)), \beta_3(m(n)), \beta_4(m(n)))$. It is sufficient that we define $\alpha_1(n)$ to be $\alpha_1(m(n))$ newly. This completes the proof of Lemma 3.

COROLLARY 4. *For any integer sequence $\alpha_1'(n)$ with $\alpha_1'(n) \to \infty$, any positive real number α, any real numbers x_3 and x_4, there exist a subsequence $\alpha_1(n)$ of $\alpha_1'(n)$ and an integer vector sequence $(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n))$ which satisfy the following five conditions.*

(1) $\beta_1\beta_2 + \beta_3\beta_4 = 2$

(2)$\lim_{n\to\infty} \dfrac{\beta_3}{\beta_1} = x_3$

(3)$\lim_{n\to\infty} \dfrac{\beta_4}{\beta_1} = x_4$

(4) β_1 and β_3 are relatively prime, and so are β_2 and β_4.

(5)$\lim_{n\to\infty} \dfrac{\beta_1}{\alpha_1} = \alpha$

PROOF: There exists a sequence $(\beta_1, \beta_2, \beta_3, \beta_4)$ which satisfies the conditions (1), (3), (4), (5) in Lemma 3 and the condition that $\lim_{n\to\infty} \dfrac{\beta_3}{\beta_1} = \dfrac{x_3}{2}$. Then, from (1) and (4), β_1 and $2\beta_3$ are relatively prime, and so are $2\beta_2$ and β_4. Thus the new sequence $(\beta_1, 2\beta_2, 2\beta_3, \beta_4)$ is a required one. Q. E. D.

Remark. Lemma 3 is a finer version of [**10**, Chap.IX, §5, Lemma] for $\pi = 2$, and Corollary 4 is for $\pi = 3$.

4. The main theorem.

Let (X, t) be a real K3 surface. We set $L_\varphi = \mathrm{Ker}(1 + t^*)$, and $L^\varphi = \mathrm{Ker}(1 - t^*)$ in $H^2(X, \mathbf{Z})$. Remark that Fix $\overline{t^*} = ((L^\varphi \otimes \mathbf{R}) \oplus i(L_\varphi \otimes \mathbf{R}))/\mathbf{R}^*$ in $P(X, \mathbf{C})$.

PROPOSITION 5. *If L_φ has $U \oplus U$ as its sublattice, then there exists a pair $\{c_1(n)\}$, $\{c_2(n)\}$ of sequences which consist of primitive elements of $U \oplus U$ and satisfy the conditions that $Q(c_1(n), c_1(n)) = Q(c_2(n), c_2(n)) = 0$, $Q(c_1(n), c_2(n)) = 2$, the sequence of the subspaces $L_n := \{\lambda \in P(X, \mathbf{C}) | Q(\lambda, c_1(n)) = Q(\lambda, c_2(n)) = 0\}$ of codimension 2 converges to a subspace $L := \{\lambda \in P(X, \mathbf{C}) | Q(\lambda, \xi_1) = Q(\lambda, \xi_2) = 0\}$ of codimension 2, where ξ_1 and ξ_2 are elements of $(U \oplus U) \otimes \mathbf{R}$, and L intersects K_{20} transversely at $H^{2,0}(X)$ in $P(X, \mathbf{C})$.*

Hence the sequence of the subspaces $L_n \cap (\mathrm{Fix}\, \overline{t^*})$ of real codimension 2 converges to the subspace $L \cap (\mathrm{Fix}\, \overline{t^*})$ of real codimension 2, and $L \cap (\mathrm{Fix}\, \overline{t^*})$ intersects $K_{20} \cap (\mathrm{Fix}\, \overline{t^*})$ transversely at $H^{2,0}(X)$ in Fix $\overline{t^*}$.

PROOF: For our sublattice of L_φ which is isomorphic to $U \oplus U$, we use the same notation $U \oplus U$. Since $U \oplus U$ is unimodular, we have $H^2(X, \mathbf{Z}) = (U \oplus U) \oplus (U \oplus U)^\perp$. Let e_1, e_2, e_3, e_4 form a basis for $U \oplus U$ and represent the intersection form Q by the matrix

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}.$$

We set $s = \mathrm{rank} L_\varphi$ and let $e_5, \ldots, e_s$ form a basis for $L_\varphi \cap (U \oplus U)^\perp$. Then $e_1, \ldots, e_s$ form a basis for L_φ. Remark that $(L_\varphi \otimes \mathbf{Q}) \oplus (L^\varphi \otimes \mathbf{Q}) = H^2(X, \mathbf{Q})$, $L_\varphi = (L^\varphi)^\perp$ and $L^\varphi = (L_\varphi)^\perp$ in $H^2(X, \mathbf{Z})$. Let $e_{s+1}, \ldots, e_{22}$ form a basis for L^φ. Then $e_1, \ldots, e_{22}$ form a basis for $H^2(X, \mathbf{Q})$. Since $H^{2,0}(X) = \overline{\iota^*}(H^{2,0}(X))$, we can take ω_X so that $\omega_X = \overline{\iota^* \omega_X}$. Then we have $\omega_X = (\sum_{j=s+1}^{22} \lambda_j e_j) + i(\sum_{j=1}^{s} \lambda_j e_j)$ for some real numbers λ_j $(1 \leq j \leq 22)$. We set $\omega_+ = \sum_{j=s+1}^{22} \lambda_j e_j$ and $\omega_- = \sum_{j=1}^{s} \lambda_j e_j$. Since $\omega_X \wedge \omega_X = 0$ and $\omega_X \wedge \overline{\omega}_X > 0$ (recall §1), we have ${\omega_+}^2 = {\omega_-}^2 > 0$. Moreover, we set $\omega'_- = \sum_{j=5}^{s} \lambda_j e_j$. Then ${\omega_-}^2 = 2(\lambda_1\lambda_2 + \lambda_3\lambda_4) + {\omega'_-}^2$. Remark that $\omega_+ \in L^\varphi \otimes \mathbf{R}$, $U \oplus U \subset L_\varphi$, where $\mathrm{sign}(U \oplus U) = (2, 2)$, and $\omega'_- \in (L_\varphi \cap (U \oplus U)^\perp) \otimes \mathbf{R}$. Since $\mathrm{sign}(H^2(X, \mathbf{Z}), Q) = (3, 19)$, we have ${\omega'_-}^2 \leq 0$. Therefore we obtain $\lambda_1\lambda_2 + \lambda_3\lambda_4 > 0$.

We may assume that $\lambda_4 \neq 0$ replacing (e_1, e_2, e_3, e_4) by (e_3, e_4, e_1, e_2) if necessary. We set

$$x_3 = \frac{\lambda_1}{\lambda_4}, \quad x_4 = \lambda_1 x_3 + \lambda_4, \quad y_4 = (1 + {x_3}^2)(\lambda_2 x_3 + \lambda_3),$$

$$\xi_1 = e_2 - x_3 e_3, \quad \xi_2 = x_3 x_4 (1 + {x_3}^2) e_1 - x_3 y_4 e_2 - y_4 e_3 + x_4 (1 + {x_3}^2) e_4.$$

We define $L = \{\lambda \in P(X, \mathbf{C}) | Q(\lambda, \xi_1) = Q(\lambda, \xi_2) = 0\}$. The subspace L meets $H^{2,0}(X)$ because $Q(\omega_X, \xi_1) = i(\lambda_1 - \frac{\lambda_1}{\lambda_4}\lambda_4) = 0$ and $Q(\omega_X, \xi_2) = i(x_3 x_4 (1 + {x_3}^2)\lambda_2 - x_3 y_4 \lambda_1 - y_4 \lambda_4 + x_4(1 + {x_3}^2)\lambda_3) = i((1 + {x_3}^2)(\lambda_2 x_3 + \lambda_3) x_4 + (-\lambda_1 x_3 - \lambda_4) y_4) = i(y_4 x_4 - x_4 y_4) = 0$. We show that L intersects K_{20} at $H^{2,0}(X)$ transversely. We identify $P(X, \mathbf{C})$ with $P^{21} = \{[X_1 : \cdots : X_{22}]\}$ taking a basis $ie_1, \ldots, ie_s, e_{s+1}, \ldots, e_{22}$. Then K_{20} is identified with the subset defined by an integral homogeneous polynomial of degree 2 of the form $f(X_1, \ldots, X_{22}) = -2(X_1 X_2 + X_3 X_4) + f_1(X_5, \ldots, X_{22})$. Hence the tangent space of K_{20} at $H^{2,0}(X)$ is identified with the subspace defined by a real linear form of the form $h(X_1, \ldots, X_{22}) = \lambda_2 X_1 + \lambda_1 X_2 + \lambda_4 X_3 + \lambda_3 X_4 + h_1(X_5, \ldots, X_{22})$. Let H denote this space. L intersects H transversely at $H^{2,0}(X)$ in P^{21}. If not, then H contains L. In particular, $(H \cap \mathbf{R}P^3 \times \{0\}) \supset (L \cap \mathbf{R}P^3 \times \{0\})$, where

$$H \cap \mathbf{R}P^3 \times \{0\} = \{\lambda_2 X_1 + \lambda_1 X_2 + \lambda_4 X_3 + \lambda_3 X_4 = 0\} \times \{0\}$$

and

$$L \cap \mathbf{R}P^3 \times \{0\}$$
$$= \{X_1 - x_3X_4 = -x_3y_4X_1 + x_3x_4(1+x_3^2)X_2 + x_4(1+x_3^2)X_3 - y_4X_4 = 0\} \times \{0\}.$$

But the following matrix is of rank 3.

$$\begin{pmatrix} \lambda_2 & 1 & -x_3y_4 \\ \lambda_1 & 0 & x_3x_4(1+x_3^2) \\ \lambda_4 & 0 & x_4(1+x_3^2) \\ \lambda_3 & -x_3 & -y_4 \end{pmatrix}$$

In fact, the determinant of the following matrix is equal to $\dfrac{2(\lambda_1^2+\lambda_4^2)^2(\lambda_1\lambda_2+\lambda_3\lambda_4)\lambda_1}{\lambda_4^5}$.

$$\begin{pmatrix} \lambda_2 & 1 & -x_3y_4 \\ \lambda_1 & 0 & x_3x_4(1+x_3^2) \\ \lambda_3 & -x_3 & -y_4 \end{pmatrix}$$

Hence, the above matrix is of rank 3 if $\lambda_1 \neq 0$. And if $\lambda_1 = 0$, then the above matrix is as follows.

$$\begin{pmatrix} \lambda_2 & 1 & 0 \\ 0 & 0 & 0 \\ \lambda_4 & 0 & \lambda_4 \\ \lambda_3 & 0 & -\lambda_3 \end{pmatrix}$$

This matrix is of rank 3 if $\lambda_1 = 0$. Thus we have a contradiction. Therefore L intersects K_{20} at $H^{2,0}(X)$ transversely.

We now show that there exists a pair $\{c_1(n)\}$, $\{c_2(n)\}$ of sequences for which the sequence $\{\lambda \in P(X, \mathbf{C}) | Q(\lambda, c_1(n)) = Q(\lambda, c_2(n)) = 0\}$ converges to the above L and the properties in the statement of Proposition 5 hold. By Corollary 4 in §3, there exists an integer vector sequence $(\alpha_{13}, \beta_{24}, -\alpha_{24}, \beta_{13})$ such that

(1) $\alpha_{13}\beta_{24} - \alpha_{24}\beta_{13} = 2$,

(2) $\lim \dfrac{-\alpha_{24}}{\alpha_{13}} = x_3$,

(3) $\lim \dfrac{\beta_{13}}{\alpha_{13}} = x_4$,

(4) α_{13} and $-\alpha_{24}$ are relatively prime, and so are β_{24} and β_{13}, and

(5) $\alpha_{13} \to \infty$.

By Lemma 3, replacing the above sequence by an appropriate subsequence if necessary, we can find an another integer vector sequence $(\alpha_{14}, \beta_{23}, -\alpha_{23}, \beta_{14})$ such that

(1′) $\alpha_{14}\beta_{23} - \alpha_{23}\beta_{14} = 1$,

(2′) $\lim \dfrac{-\alpha_{23}}{\alpha_{14}} = 0$,

(3′) $\lim \dfrac{\beta_{14}}{\alpha_{14}} = y_4$, and

(4′) $\lim \dfrac{\alpha_{14}}{\alpha_{13}} = \dfrac{1}{\sqrt{2}}$.

We set

$$\alpha_1 = \alpha_{13}\alpha_{14}, \quad \alpha_2 = \alpha_{23}\alpha_{24}, \quad \alpha_3 = -\alpha_{13}\alpha_{23}, \quad \alpha_4 = \alpha_{14}\alpha_{24},$$

$$\beta_1 = \beta_{13}\beta_{14}, \quad \beta_2 = \beta_{23}\beta_{24}, \quad \beta_3 = -\beta_{13}\beta_{23}, \quad \beta_4 = \beta_{14}\beta_{24}.$$

Then we have

$$\alpha_1\alpha_2 + \alpha_3\alpha_4 = \beta_1\beta_2 + \beta_3\beta_4 = 0$$

and

$$\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_3 = (\alpha_{13}\beta_{24} - \alpha_{24}\beta_{13})(\alpha_{14}\beta_{23} - \alpha_{23}\beta_{14}) = 2.$$

From (4) and (1′) above, we see that $\alpha_1, \alpha_2, \alpha_3$ and α_4 are relatively prime. So are $\beta_1, \beta_2, \beta_3$ and β_4. Hence, if we set $c_1 = \alpha_1 e_2 + \alpha_2 e_1 + \alpha_3 e_4 + \alpha_4 e_3$ and $c_2 = \beta_1 e_2 + \beta_2 e_1 + \beta_3 e_4 + \beta_4 e_3$, then $Q(c_1(n), c_1(n)) = Q(c_2(n), c_2(n)) = 0$, $Q(c_1(n), c_2(n)) = 2$, and moreover, $c_1(n)$ and $c_2(n)$ are primitive elements in $U \oplus U$ (hence in $H^2(X, \mathbf{Z})$).

Finally we show that the sequence $L_n = \{Q(\lambda, c_1(n)) = Q(\lambda, c_2(n)) = 0\}$ converges to L. We first observe that

$$\lim \frac{\alpha_2}{\alpha_1} = \lim \frac{\alpha_{24}}{\alpha_{13}} \lim \frac{\alpha_{23}}{\alpha_{14}} = (-x_3) \cdot 0 = 0,$$
$$\lim \frac{\alpha_3}{\alpha_1} = \lim \frac{-\alpha_{23}}{\alpha_{14}} = 0,$$
$$\lim \frac{\alpha_4}{\alpha_1} = \lim \frac{\alpha_{24}}{\alpha_{13}} = -x_3,$$
$$\lim \frac{\beta_2}{\beta_1} = \lim \frac{\beta_{24}}{\beta_{13}} \lim \frac{\beta_{23}}{\beta_{14}} = (-x_3) \cdot 0 = 0,$$
$$\lim \frac{\beta_3}{\beta_1} = \lim \frac{-\beta_{23}}{\beta_{14}} = 0,$$

and

$\lim \dfrac{\beta_4}{\beta_1} = \lim \dfrac{\beta_{24}}{\beta_{13}} = -x_3.$

Hence both $[\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4]$ and $[\beta_1 : \beta_2 : \beta_3 : \beta_4]$ converge to $[1 : 0 : 0 : -x_3]$. Thus both $\{Q(\lambda, c_1(n)) = 0\}$ and $\{Q(\lambda, c_2(n)) = 0\}$ converge to $\{Q(\lambda, \xi_1) = 0\}$. In order to know the limit subspace of $\{L_n\}$, we set

$$B_j = (\sum_{i=1}^{4} \alpha_i{}^2)\beta_j - (\sum_{i=1}^{4} \alpha_i\beta_i)\alpha_j \ (j = 1, 2, 3, 4).$$

Remark that (B_1, B_2, B_3, B_4) are orthogonal to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in $\mathbf{R}^4$ with respect to the Euclidean inner product. We set

$$\tilde{c}_2 = B_1e_2 + B_2e_1 + B_3e_4 + B_4e_3.$$

Then we see $L_n = \{Q(\lambda, c_1(n)) = Q(\lambda, \tilde{c}_2(n)) = 0\}$. We now consider the limit hyperplane of the sequence $\{Q(\lambda, \tilde{c}_2(n)) = 0\}$. Since

$B_1 = \alpha_2(-2\alpha_{23}\beta_{14} - \alpha_{13}\beta_{24}) + \alpha_3\alpha_{13}\beta_{13} - 2\alpha_4\alpha_{14}\beta_{14},$

$B_2 = \alpha_1(2\alpha_{23}\beta_{14} + \alpha_{13}\beta_{24}) - 2\alpha_3\alpha_{23}\beta_{23} + \alpha_4\alpha_{24}\beta_{24},$

$B_3 = \alpha_4(2\alpha_{14}\beta_{23} - \alpha_{13}\beta_{24}) - \alpha_1\alpha_{13}\beta_{13} - 2\alpha_2\alpha_{23}\beta_{23}$

and

$B_4 = \alpha_3(-2\alpha_{14}\beta_{23} + \alpha_{13}\beta_{24}) + 2\alpha_1\alpha_{14}\beta_{14} - \alpha_2\alpha_{24}\beta_{24};$

we have

$\lim \dfrac{B_1}{\alpha_1{}^2} = \sqrt{2}x_3y_4,$

$\lim \dfrac{B_2}{\alpha_1{}^2} = -\sqrt{2}x_3x_4(1 + x_3{}^2),$

$\lim \dfrac{B_3}{\alpha_1{}^2} = -\sqrt{2}x_4(1 + x_3{}^2)$

and

$\lim \dfrac{B_4}{\alpha_1{}^2} = \sqrt{2}y_4.$

Hence

$[B_1 : B_2 : B_3 : B_4]$ converges to $[-x_3y_4 : x_3x_4(1 + x_3{}^2) : x_4(1 + x_3{}^2) : -y_4]$. Namely, $\{Q(\lambda, \tilde{c}_2(n)) = 0\}$ converges to $\{Q(\lambda, \xi_2) = 0\}$. Therefore L_n converges to L. With respect to the identification $P(X, \mathbf{C}) \simeq P^{21}$ stated above, $\mathbf{R}P^{21}$ corresponds to Fix $\overline{t^*} = (i(L_\varphi \otimes \mathbf{R}) \oplus (L^\varphi \otimes \mathbf{R}))/\mathbf{R}^*$. Hence the latter assertion of the proposition follows. Q. E. D.

We next consider a family (V, M, π) of complex structures of X with antiholomorphic involutions t_V and t_M, and the period mapping $\tau : M \to P(X, \mathbf{C})$ as stated in Kharlamov's lemma (recall §1).

Theorem 6. *Let (X,t) be a real K3 surface. If L_φ has $U \oplus U$ as its sublattice, then there exist points α in Fix t_M for which real K3 surfaces $(V(\alpha), t_\alpha)$ can be 2-sheeted coverings of $P^1 \times P^1$ (Let Φ_α denote the covering maps.) branched along nonsingular curves defined by real homogeneous polynomials of degree (4,4) and satisfy $conj \circ \Phi_\alpha = \Phi_\alpha \circ t_\alpha$ arbitrarily closely to m.*

Proof: We set $(U \oplus U)_\alpha = i_\alpha^* \circ i_m^{*-1}(U \oplus U)$ for any α in M. The isomorphisms $i_\alpha^* \circ i_m^{*-1} : H^2(X, \mathbf{Z}) \to H^2(V(\alpha), \mathbf{Z})$ preserve the intersection forms. Let Q_α denote the intersection form on $V(\alpha)$. Recall that we set $t_\alpha = t_V|_{V(\alpha)}$ for every α in Fix t_M. We set $L_\alpha = \mathrm{Ker}(1 + t_\alpha^*)$ in $H^2(V(\alpha), \mathbf{Z})$. Since $L_\alpha = i_\alpha^* \circ i_m^{*-1}(L_\varphi)$, we have $(U \oplus U)_\alpha \subset L_\alpha$. Let $\{L_n\}$ be a sequence obtained by Proposition 5. Then for a sufficiently large natural number N, $L_n \cap \mathbf{R}P^{21}$ intersects $\tau(\text{Fix } t_M) = K_{20} \cap \mathbf{R}P^{21}$ transversely at $H^{2,0}(X)$ in $\mathbf{R}P^{21} = (i(L_\varphi \otimes \mathbf{R}) \oplus (L^\varphi \otimes \mathbf{R}))/\mathbf{R}^*$ (recall the proof of Proposition 5) for any $n \geq N$. Hence $L_n \cap \tau(\text{Fix}t_M)$ is nonempty and real 18 dimensional. We set

$$\hat{E} = \{\tau(\alpha) \in \tau(M) | \text{rank Pic}V(\alpha) \geq 3\}.$$

From the results in [**10**, Chap.IX, §4, p.215], rank Pic$V(\alpha) \geq 3$ if and only if $Q(\tau(\alpha), c_j^\alpha) = 0$ for elements c_j^α $(j = 1,2,3)$ in $H^2(X, \mathbf{Z})$ which are linearly independent over $\mathbf{C}$ (hence, over $\mathbf{R}$). Hence $L_n \cap \tau(\text{Fix } t_M) \cap \hat{E}$ can be covered by countably many real 17 dimensional submanifolds. Hence $(L_n \cap \tau(\text{Fix } t_M)) \setminus \hat{E}$ is dense in $L_n \cap \tau(\text{Fix } t_M)$, and for every $\tau(\alpha) \in (L_n \cap \tau(\text{Fix } t_M)) \setminus \hat{E}$, we have $\alpha \in$Fix t_M and rank Pic$V(\alpha) = 2$. We set $c_{j\alpha}(n) = i_\alpha^* \circ i_m^{*-1}(c_j(n))$ for j $(= 1, 2)$. Then $Q_\alpha(c_{1\alpha}, c_{1\alpha}) = Q_\alpha(c_{2\alpha}, c_{2\alpha}) = 0$ and $Q_\alpha(c_{1\alpha}, c_{2\alpha}) = 2$. Since $Q(i_m^* \circ i_\alpha^{*-1}(H^{2,0}(V(\alpha))), c_j) = 0$, we have $Q_\alpha(H^{2,0}(V(\alpha))), c_{j\alpha}) = 0$, that is, $c_{j\alpha} \in$ Pic$V(\alpha) = (H^{2,0}(V(\alpha))^\perp) \cap H^2(V(\alpha), \mathbf{Z})$. We see that $c_{1\alpha}$ and $c_{2\alpha}$ are primitive elements in $(U \oplus U)_\alpha$, hence in $H^2(V(\alpha), \mathbf{Z})$. Recall that $(U \oplus U)_\alpha \subset L_\alpha = \mathrm{Ker}(1 + t_\alpha^*)$. Hence $(V(\alpha), t_\alpha)$ satisfies the conditions of Lemma 2. Since $(L_n \cap \tau(\text{Fix } t_M)) \setminus \hat{E}$ is dense in

$L_n \cap \tau(\text{Fix } t_M)$ and n $(\geq N)$ is an arbitrary number, we can choose such $\alpha \in \text{Fix } t_M$ arbitrarily closely to m. This completes the proof of Theorem 6.

COROLLARY 7. *Let (X,t) be a real K3 surface. If L_φ has $U \oplus U$ as its sublattice, then there exists a 2-sheeted covering $\Phi : Y \to P^1 \times P^1$ branched along a nonsingular real curve of degree (4,4) and an antiholomorphic involution T on Y such that $conj \circ \Phi = \Phi \circ T$ and Fix T is diffeomorphic to Fix t.*

PROOF: We can consider the restriction $\mathbf{R}\pi : \text{Fix } t_V \to \text{Fix } t_M$ of the family (V, M, π). Although Fix t_M is possibly disconnected, we may consider that α of Theorem 6 and m are contained in the same connected component of Fix t_M. Since $\mathbf{R}\pi$ is a proper submersion onto Fix t_M, $\mathbf{R}\pi^{-1}(\alpha)$ is diffeomorphic to $\mathbf{R}\pi^{-1}(m)$, where $\mathbf{R}\pi^{-1}(\alpha) = \text{Fix } t_\alpha$ and $\mathbf{R}\pi^{-1}(m) = \text{Fix } t$. It is sufficient to set $Y = V(\alpha)$ and $T = t_\alpha$. Q. E. D.

COROLLARY 8. *Three possible configuration types $\frac{1}{1}8$, $\frac{5}{1}4$ and $\frac{9}{1}$ are all realized by some real curves of degree (4,4).*

PROOF: As stated in §0, there exist real projective K3 surfaces (X,t) with $h^2 = 4$ (h : primitive) whose real parts are homeomorphic to $\Sigma_{10} \coprod S^2$, $\Sigma_6 \coprod 5S^2$ and $\Sigma_2 \coprod 9S^2$ respectively. Moreover, for such real K3 surfaces, L_φ are isomorphic to $U \oplus U \oplus (-E_8) \oplus (-E_8)$, $U \oplus U \oplus (-E_8)$ and $U \oplus U$ respectively (see [8]). Hence L_φ have $U \oplus U$ as sublattices. By Corollary 7 and [5, §3] (recall §0), we obtain our required results. Q. E. D.

REFERENCES

1. P. Griffiths and J. Harris, "Principles of Algebraic Geometry," John Wiley & Sons, New York, 1978.

2. D.A. Gudkov, *The topology of real progective algebraic manifolds*, Uspekhi Mat. Nauk = Russian Math. Surveys **29** (1974), 1–79.
3. V.M. Kharlamov, *The topological types of nonsingular surfaces of degree 4 in* $\mathbf{R}P^3$, Funktsional'ni Analiz i ego Prilozheniya = Functional Anal. Appl. **10** (1976), 295–305.
4. K. Kodaira, *On compact complex analytic surfaces. I*, Ann. of Math. (2) **71** (1960), 111–152.
5. S. Matsuoka, *Nonsingular algebraic curves in* $\mathbf{R}P^1 \times \mathbf{R}P^1$, Trans. Amer. Math. Soc. (to appear).
6. A.L. Mayer, *Families of K-3 surfaces*, Nagoya Math. J. **48** (1972), 1–17.
7. D. Mumford, "Algebraic Geometry I, Complex Projective Varieties," Springer-Verlag, 1976.
8. V.V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Izv. Akad. Nauk SSSR = Math. USSR Izv. **14** (1980), 103–167.
9. I.I. Pjateckii-Shapiro and I.R. Shafarevich, *The arithmetic of K3 surfaces*, Trudy Mat. Inst. Steklov = Proc. Steklov Inst. Math. **132** (1973), 45–57.
10. I.R. Shafarevich et al., *Algebraic surfaces*, Trudy Mat. Inst. Steklov = Proc. Steklov Inst. Math. **75** (1965).
11. J.J. Wavrik, *Deformations of Banach coverings of complex manifolds*, Amer. J. Math. **90** (1968), 926–960.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060, JAPAN